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Risk Management Models

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Abstract

Financial risk management models were often used wrongly prior to the 2007 credit crisis, and they are still being used wrongly today. This misuse contributed to the crisis. We show that there are two common misuses of derivative pricing models associated with calibration and hedging "the greeks." The purpose of this paper is to clarify these misuses and explain how to properly use risk management models. In particular we show that: (i) the implied default probabilities from structural credit risk models and the default probabilities obtained from credit risk copula models are misspecified, and (ii) vega hedging is a nonsensical procedure.

1 Introduction

The 2007 credit crisis was a wake-up call with respect to model usage. It has been alleged that the misuse of risk management models helped to generate the crisis. For example, "*The turmoil at AIG is likely to fan skepticism about the complicated, computer-driven modeling systems that many financial giants rely on to minimize risk.*"[1] Even the financial wizard himself, Warren Buffet, is skeptical about models "*All I can say is, beware of geeks...bearing formulas.*"[2] Unfortunately for financial engineers, these allegations are true. In numerous cases, models were used wrongly by the financial industry prior to the 2007 credit crisis, and they continue to be used wrongly today. Unfortunately, we cannot just blame the financial industry for this faulty misuse. These techniques are often presented without adequate explanation in the standard textbooks on derivatives (see Chance [5], Hull [10], Jarrow and Turnbull [15], Kolb [16], Whaley [20], Rebonato [18]).

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[1]Wall Street Journal, Monday Nov 3, 2008, *Behind AIG's Fall, Risk Models Failed to Pass Real-World Test.*

[2]Wall Street Journal, Monday Nov 3, 2008, *Behind AIG's Fall, Risk Models Failed to Pass Real-World Test.*

The purpose of this paper is to critically analyze risk management model construction, testing, and usage. In constructing models, we characterize a model's assumptions as being one of two types: robust or critical. Robust assumptions are those which if changed slightly, only change the model's results slightly as well. Critical assumptions are those for which this is not the case. In testing models we argue that one needs to test the model's implications and its critical assumptions. If either of these are rejected, the model should not be used. With respect to model implementation, we show that there are two common misuses of derivative valuation models related to calibration and hedging of the greeks. With respect to calibration, we show that the implied default probabilities from structural credit risk models and the default probabilities obtained from credit risk collateralized debt obligation (CDO) copula models are misspecified. In addition, with respect to hedging of the greeks, we show that the commonly used vega hedge is a nonsensical procedure.

An outline of this paper is as follows. Section 2 presents a characterization of models and explains how models should be tested. Section 3 analyzes the two common misuses of risk management models, section 4 synthesizes the previous insights with a discussion on how to properly use models, and section 5 concludes.

2 Models as Approximations

Statistician George E.P. Box ([3], p.74) said "Remember that all models are wrong: the practical question is how wrong do they have to be to not be useful." George Box is correct. All models are approximations, and as such, they are formally incorrect. However, this should not preclude a model's use. A model's usefulness - the quality of the approximation - needs to be judged relative to a purpose or an "objective function." Very crude models can be useful for decision making in some contexts, whereas more sophisticated models are needed in others. But, before we can continue, we need to first understand what a model is.

2.1 What is a Model?

Consider the following diagram.

$$\left[Model \iff \begin{cases} assumption\ 1 \\ assumption\ 2 \end{cases} \Leftrightarrow \begin{cases} implication\ 1 \\ implication\ 2 \end{cases} \right]$$

Figure 1: Abstract Representation of a Model

A model is equivalent to its assumptions. This is by construction, a tautology. Equivalently, if one generates enough implications - both necessary and sufficient - then the model is equivalent to its implications. In practice, it is very difficult to identify all the necessary implications of a model thereby making them a sufficient set of implications. For simplicity, we have indicated only two (sets of) assumptions and two (sets of) implications.

With respect to the model's assumptions, there are two opposing views on how to test a model.

Milton Friedman [9] said "a theory cannot be tested by comparing its "assumptions" directly with "reality." Indeed, there is no meaningful way in which this can be done....the question of whether a theory is realistic 'enough' can be settled only by seeing whether it yields predictions that are good enough for the purpose in hand or that are better than predictions from alternative theories." More recently, Robert Aumann ([1], p.388) also echoes these views.

In contrast, Robert Solow [19] said "All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive. A 'crucial' assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic. When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect."

To understand which view is correct, we first need to step back and discuss a model's assumptions in more detail.

2.2 Two Types of Assumptions

There are two types of assumptions with respect to any model: robust and non-robust (critical assumptions). A *robust assumption* is one where the implications of the model only change slightly if the assumption is modified only slightly. This corresponds to a "continuity" in the topology of the model's structure. In contrast, a non-robust or critical assumption is one where the implications of the model change discretely if the assumption is only changed slightly. This corresponds to a "discontinuity" in the topology of the model's structure. Hence, we can revise our diagram about models.

$$\left[Model \iff \left\{ \begin{array}{c} robust\ assumptions \\ critical\ assumptions \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} implication\ 1 \\ implication\ 2 \end{array} \right. \right]$$

Figure 2: Abstract Representation of a Model including the Robust and Critical Assumptions

This distinction is important because since models are approximations of a complex reality, we may not get the assumptions exactly correct. With robust assumptions, we do not need to worry too much. For small errors in the robust assumptions, the implications only change by a small amount. We need to be careful, however, with the critical assumptions. If we get the critical assumptions wrong, by just a little, the implications completely change. Of course, for large errors in robust assumptions, the implications will also change significantly, despite the "continuity." So, even the robust assumptions are important in model construction.

There is only one way to determine if an assumption is critical or robust. This is to extend the model, relax the assumptions, and determine analytically if the implications are changed continuously or not. This determination is a key purpose of generalizing models.

Example 1 ***Derivative Pricing Model Assumptions.***

The standard derivative pricing models assume frictionless markets (no transaction costs, no trading restrictions or short sale constraints), competitive markets, an evolution for the stock price process, and no arbitrage. With respect to these assumptions, we can partition them into robust and critical assumptions as follows:

- *No transaction costs, and temporary quantity impacts on the price process are robust assumptions (see Broadie, Cvitanic and Soner [4], Jarrow and Protter [14]). A small transaction cost or temporary quantity impact on the price will not change pricing nor hedging dramatically. These frictions create a wedge around the frictionless price.*

- *No trading restrictions or short sale constraints is a critical assumption. Its violation makes the market incomplete and eliminates certain hedging strategies. Pricing changes completely as a result. An extreme form of this assumption is that the underlying asset doesn't trade, which implies hedging is impossible and the market is incomplete.*

- *Competitive markets (or no permanent quantity impacts on the price process) is a critical assumption. Violations of this assumption generate market manipulation. Market manipulation causes a complete reversal of pricing and hedging (see Jarrow [11]).*

- *The assumed stock price process is a robust assumption as long as a small perturbation is defined as one which retains market completeness. As is well known, changing from a complete to an incomplete market dramatically changes pricing and hedging. Such a change in a stock price process can be viewed as non-local change in a robust assumption.*

- *No arbitrage is a critical assumption. If a market contains an arbitrage opportunity, then market prices need not satisfy any of the implications of the derivative valuation model.*

2.3 Testing Models

Now, we can move back to the testing of models. To test a model, the usual method is to first look at the implications. If one tests implication 1, and accepts it, this does not mean the model is true. To prove the model, we need to also accept implication 2. But, the typical situation is that we cannot test implication 2 (perhaps there is no observable data on implication 2).[3] And, we often want to use implication 2 in some application. Implication 2 may, in fact, be the reason why the model was developed. So, the question then arises, should one use the model for implication 2?

Inconsistent with Milton Friedman's view, the answer is that it depends on the assumptions. Logically, there is another way to test a model. Instead of testing all the implications, one can directly test the assumptions themselves. One can reject the model if any assumption is false; and, one can accept a model only if every assumption is true.

Now, back to the question, should one use the model? The answer is no, unless some additional conditions are satisfied. First, consistent with to Robert Solow's views, one needs to be sure that the critical assumptions are valid. Otherwise, if a critical assumption is false, we know (logically) that implication 2 is false, and significantly so (due to the discontinuity in the model's implications). And, this is true even though implication 1 holds. Second, one needs to be sure that the robust assumptions are approximately true. Even a robust assumption, if grossly incorrect, will lead to implications which are significantly off.

Thus, the testing of a model requires a testing of the assumptions. In this regard, the construction of the model is important. A good model is one that is constructed such that the critical assumptions are observable and can be tested. For the robust assumptions, these can be more abstract, not directly observable nor testable. This distinction in the ability to test the assumptions is what differentiates a "good" model construction for a "bad" model construction.

Thus, the model should be used for implication 2, only if:

(i) implication 1 is tested and it appears valid,

(ii) all the critical assumptions are tested and they appear valid, and

(iii) all of the robust assumptions are approximately true (and accepted without testing).

If any of (i), (ii) or (iii) are violated, then the model should be rejected and not used.

Example 2 ***A Lucky Charm.***

A simple example illustrates the logical error of using a model if an implication appears true, but the underlying assumption is not.

Suppose that a baseball player wears a pair of socks and gets a home run. He wears the socks again, and he hits another home run. This becomes his "lucky charm."

The model is "wearing the socks generates a home run." The player observes the implication "a home run" is true, and accepts the model. This is testing a model based on only the implications being satisfied (Friedman's view).

My approach would be to look at the critical assumption itself, "wearing the sock generates." As there is no scientific evidence for the satisfaction of this assumption - as it is not true, I would reject the model (Solow's view).

[3] Another reason could be that implication 2 is not known - we do not have necessary and sufficient implications.

Although simple, this example can also be used to show the consequences of using such a lucky charm model in industry. The continued coincidence of a successful outcome and the application of the model gives the user increased over-confidence in the model's validity. If the other implications of the model are then used as well, problems can arise. At some point the luck will run out. This analogy applies to the credit rating agencies and AIG's use of the copula model for gauging a Credit Debt Obligations (CDO) default risk (see Bluhm and Overbeck [2] and the discussion below on calibration). A related and continuing example of the misuse of credit risk models is with respect to the debate between using structural versus reduced form models (see Jarrow [13] for a recent discussion of this debate).

Example 3 ***Credit Risk Models.***

In credit risk modeling there are two model types: reduced form and structural. A crucial assumption in each is the ability to trade the underlying assets (see Jarrow [12] for a review). A violation of this assumption causes the implications of the model to completely change. This is a violation of the no trading restrictions assumption of derivative pricing models discussed earlier.

For the reduced form models the traded assets are the firm's risky debt. For structural models the traded assets are all of the firm's assets (equivalently, all of the firm's liabilities and equity).

One can test this critical assumption for both models by direct observation. For reduced form models it is accepted, for structural models it is rejected. Hence, I reject the structural models based on this evidence alone.[4]

In industry practice, both model types are used to infer a firm's probability of default for risk management purposes. The structural models, being rejected by direct observation, should not be used for this purpose.

3 Two Common Option Model Misuses

Now that we understand how to test a model for its usefulness, we turn to the use of models. Models are often misused. This can sometimes occur in subtle ways. With respect to the pricing and hedging of derivatives, calibration and hedging of the greeks are two common model misuses. In fact, these common misuses were contributing causes of the 2007 credit crisis. Unfortunately, we cannot just blame the financial industry for this faulty misuse. These techniques are also often presented without adequate explanation in the standard textbooks on derivatives.

[4] Of course, various scholars have tested implications of both models. In this regard, the evidence rejects the implications of simple forms of the structural model and accepts the reduced form models (see Jarrow [13] for a review). This has lead various authors to seek generalizations of the simple structural model whose implications are more consistent with the evidence. However, these generalizations still maintain the same critical assumption, which can be directly tested and rejected.

3.1 Two Types of Models

First, we need to step back and decompose models into two types: (1) statistical and (2) theoretical. A statistical model identifies patterns in historical data that one hopes will continue into the future. A theoretical model is based on economic reasoning - whose parameters are estimated from historical data.

We will talk as if this distinction is black and white, but as one might expect, it is really more gray (given that the inputs of a theoretical model are also estimated statistically). Nonetheless, this distinction will help us to understand the issues involved. To help explain the difference between these two types of models, it is easiest to use an example from derivatives pricing to illustrate the differences.

Example 4 ***Call Option Valuation***.

Suppose that the purpose of the model is to price a European call option on a stock. We illustrate both a theoretical and statistical model useful for this purpose.

The Black-Scholes model is a theoretical model for the call's price. Its parameters are estimated using historical asset prices. Generalizations of the Black-Scholes model for stochastic volatilities and jumps also exist. These are all theoretical models.

A simple statistical model for pricing options can be generated as follows. Using historical option prices, compute the average call price conditioned on the strike price, the time to maturity, the stock's volatility, and the level of the stock price to obtain an estimate of the option's value. A more sophisticated version of this approach may use time series estimation methods.

Both models are used in practice today. As argued below, a more sophisticated class of statistical models are those that develop time series models (e.g. GARCH) to estimate (predict) an option's implied volatilities.

Both types of models are useful. They have different costs and benefits. The benefits of theoretical models are that they attempt to understand the causes of the implications, but they are often hard to get right. The benefits of statistical models are that they are often easy to construct, but they do not explain the underlying cause of the pattern. Of course, if the pattern changes, then the statistical model will no longer be useful. If the theoretical model captures this cause, then it will still apply.

Given this decomposition in model types, we can now discuss calibration and hedging the greeks.

3.2 Calibration

Calibration is defined to be the estimation of a theoretical model's parameters by equating the model's price (or more generally, its implications) to market prices (or more generally, to market observables). This section shows that calibration transforms a theoretical model into a statistical model. However, a calibrated risk management model is commonly and incorrectly still believed to be a theoretical model. This incorrect belief can lead to misuse as illustrated below.

3.2.1 What is calibration?

To define calibration, we start with a theoretical model $f : R^2 \to R$ representing a security's price, i.e.

$$p_u = f(q, u)$$

where u is a known constant (or vector) of parameters, q is the parameter to be estimated, and p_u is the market price. To help with the intuition, one can think of p_u as a call option's price, u represents the strike price and time to maturity, q is the underlying stock's volatility, and $f(q, u)$ is the Black-Scholes model. Although, the formulation is purposely much more general and abstract. To use the theoretical model, one estimates q directly using historical data. This gives the estimate: $\widehat{q}$.

Suppose, however, that in the process of this estimation, one discovers that

$$p_u^{obs} \neq f(\widehat{q}, u)$$

where p_u^{obs} corresponds to the observed market price. This observation rejects the theoretical model. Nonetheless, one still wants to use the model. Is this still possible? The answer is yes. This is the purpose of calibration.

Here is how calibration works. Given is some set of market data $\{p_1^{obs}, u_1\}$. Using this data, one estimates q^{cal} such that $p_1^{obs} = f(q^{cal}, u_1)$, i.e.

$$q^{cal} = f^{-1}(p_1^{obs}, u_1).$$

Then, given this estimate for q^{cal}, the model for the security's price p_u for $u \neq u_1$ is now estimated by

$$p_u = f(f^{-1}(p_1^{obs}, u_1), u).$$

As represented, it is now easy to see that this is a statistical model based on a non-linear estimation. It is a model based on matching patterns in historical data. Although the choice of the functional form for the non-linear estimation is based on theory, the theoretical model itself is rejected. Hence, it is no longer a theoretical model. This simple observation is not well understood.

3.2.2 When should calibration be used?

Given that we have determined that calibration is a statistical model formulated to match market prices p, it should be used for no other purpose. This follows precisely because the need for calibration is predicated on the fact that its use as a theoretical model has been rejected. Using the calibrated model to obtain the additional implications of the theory is no longer valid.

The argument can be understood by referring to Figure 2. By construction, the need to calibrate is predicated on the theoretical model's rejection when using historical data to estimate the model's parameters. Next, a (non-linear) statistical model is generated by calibration. The statistical model is constructed to guarantee that an implication - pricing - is valid. It is therefore OK to use the calibrated model for this implication - pricing, since the model is validated for that use.[5] The problem occurs when trying to use the calibrated model for another implication, for example hedging. By simple logic, since the model is rejected, this second implication does not follow. The "bottom line"of this reasoning is that *the calibrated model is no longer valid and does not imply the implications of the original theoretical model.* Hence, it is wrong to use a calibrated pricing model for any other implication besides pricing.

Example 5 ***Implied Black-Scholes Volatilities.***

The first example of calibration and its misuse is with respect to the Black-Scholes model and implied volatilities.

It is well-known (see Jarrow and Turnbull [15]) that for equity options, when using historical volatility estimates, one can reject the Black-Scholes model.

Hence, to match market prices, calibration is used. Given the key unknown is the equity's volatility, the use of implied volatilities has arisen. When estimating implied volatilities, a smile or sneer in the strike price and the time to maturity profile has been observed.[6]

These implied volatilities are useful as a statistical model for estimating option prices. One can use sophisticated time series methods in this regard (e.g. GARCH). The misuse occurs, however, if one still attempts to use the Black-Scholes model with implicit volatilities to do hedging (see Chance [5], Hull [10], Kolb [16], Whaley [20]). This use is invalid because the theoretical Black-Scholes model has been rejected.

Not surprisingly, in this use, it has been observed that delta (plus gamma) hedging doesn't work. Consequently, the practice of vega hedging has arisen (vega hedging is another misuse of risk management models that is separately discussed below).

These same incorrect procedures are also common market practice when pricing caps and floors using a calibrated LIBOR market model, see Rebonato [18].

Example 6 ***Implied Default Probabilities using Merton's ([17]) Model.***

A second example of the misuse of calibration and its misuse is with respect to structural credit risk models and implied probabilities of default.

As discussed previously, it is well-known that structural credit risk models assume that all of the firm's assets trade and their values are observable (see Jarrow [13]). It is also an accepted fact that the firm's assets do not trade and the firm's value (and volatility) are unobservable. These facts reject the structural model's critical assumptions, and hence the model.

Nonetheless, the model is still desired to be used. To estimate these inputs, calibration is used. The calibration uses observable stock prices and the stock price's volatility to obtain estimates of the firm's value and volatility. This is often done my minimizing the error between the market observed stock price and

[5] Of course, one needs to design the use of the calibrated model so that it is not based on circular reasoning.

[6] Of course, this observation also provides a further rejection of the Black-Scholes model.

volatility versus the model's values. As a result of this estimation, the calibrated structural model is now useful for obtaining a non-linear estimate of the stock's price and volatility.

However, in industry practice, the model is still incorrectly used as a theoretical model for a second purpose. The second purpose is to estimate the firm's probability of default (or the distance to default). The probability of default estimates obtained from this procedure are invalid precisely because the calibration is predicated on the model's rejection. Since the model is rejected, these default probability estimates are misspecified and should not be used.[7]

Example 7 ***Implied Default Correlations using a CDO Copula Model.***

A third example of the misuse of calibration is with respect to the copula model for pricing collateralized debt obligations (CDOs) and implied default correlations. The standard CDO copula pricing model (see Bluhm and Overbeck [2]) depends on the correlations between the underlying firm's asset returns (pairwise). For simplicity, it is often assumed that these correlations are the same across all firms, called ρ.[8]

As with the structural model, the CDO copula model assumes that the assets of the firm trade and are observable. As discussed in the previous example, these assumptions are not true and the hence the model is formally rejected.[9]

Nonetheless, the model is still desired to be used to price CDOs. For this purpose, an implied correlation coefficient can be obtained by calibrating the CDO's model price to the market price. When calibrated in this manner, the CDO copula model generates a valid statistical model for pricing CDO's.

The misuse occurs when the model is still viewed as a theoretical model and employed for a second purpose. The second purpose is to estimate the probability distribution for the losses of the CDO's underlying collateral pool. Since the theoretical CDO copula model is rejected, its use to estimate this loss distribution is invalid.

Unfortunately, it was this misuse that contributed to the incorrect belief prior to the 2007 credit crisis that subprime CDOs were safe. This misuse was the basis of the credit rating agencies incorrect ratings of subprime CDOs (see Crouhy, Jarrow, Turnbull [6]).

3.3 Hedging the Greeks

Another misuse of derivative pricing models is related to "hedging the greeks." Some derivative model's greeks can be hedged correctly and some cannot. The greeks that can be hedged correctly are related to stochastic processes as they represent risks within the derivative model. The greeks that are hedged incorrectly are those that are constant or deterministic parameters and that do not

[7] There is an abundance of academic evidence rejecting these default probabilties as good estimates of the true underlying default probabilities (see Jarrow [13]).

[8] Some minor generalizations of this simplifying assumption have been considered (see Jarrow [12]). Nonetheless, the logic as discussed here still applies.

[9] Of course, since the assets are unobservable, we cannot estimate this correlation directly.

represent risks within the model. For example, in the Black-Scholes model, the stock price is a correct greek to hedge (the delta and gamma), but the volatility (the vega) or the interest rate (the rho) are not.

It is easiest to explain this misuse using an example that is in most textbooks (see Chance [5], Hull [10], Kolb [16], Whaley [20], Rebonato [18]) and that is quite common in industry usage: vega hedging or hedging the changing option's volatility when using the Black-Scholes formula.

3.3.1 What is Vega hedging?

Vega hedging is based on the Black-Scholes model. Let $p = f(S, \sigma)$ represent the Black-Scholes model where σ is the stock's volatility, assumed constant, p is the option's price, and S is the stock price. Here only S is random, hence, there is only one risk in the model. This price risk is embedded within the model via the standard complete market hedging argument and the use of risk-adjusted (or risk neutral) probabilities (see Jarrow and Turnbull [15]).

Hence, as is well known, if the model is correct, then to hedge the option's risk, we need to only hedge changes in S using the stock's delta. However, if we cannot trade quickly enough, gamma hedging is appropriate to hedge changes in the delta as well. As these hedges relate to price risk, which is incorporated into the model's construction, this usage is correct.

But, in practice and in contradiction to the model's assumptions, we observe that the stock's volatility σ is random. And, we observe that delta and gamma hedging are insufficient to remove the risk from a hedged stock and option position. Of course, this formally rejects the model.

Nonetheless, the Black-Scholes model is still used in an attempt to hedge the volatility risk. To "fix" this hedging error, a Taylor series expansion is applied to the Black-Scholes formula writing the change in the option's price as a linear function of the option's delta times ΔS plus the option's vega $\frac{\partial f}{\partial \sigma}$ times $\Delta \sigma$. The goal is to use this linear expansion of the Black-Scholes formula to hedge changes in the volatility ($\Delta \sigma$).

But, this linear approximation is nonsensical because the volatility is a constant ($\Delta \sigma \equiv 0$) in the model's construction. There is no volatility risk priced within the Black-Scholes model, so the vega ($\frac{\partial f}{\partial \sigma}$) can give no indication of how changes in the volatility affect the call's price. This is the common fallacy. Vega hedging is thus a nonsensical procedure.

We can understand the magnitude of the error introduced in vega hedging by considering a more general call pricing model that includes both risks.

Example 8 ***Vega Hedging.***

This example shows the magnitude of the errors that occur in vega hedging when using the Black-Scholes formula.

Let us consider a more general stochastic volatility option pricing model where σ_t is the stock's stochastic volatility at time t. It can be shown (see Eisenberg and Jarrow [7], Fouque, Papanicolaou and Sircar [8]) that the true *call option price is*

$$C = \int_0^\infty f(\Theta_0) h(\Theta_0 \mid \sigma_0) d\Theta_0$$

where $f(\cdot)$ *is the BS formula,* T *is the option's maturity date,* $\Theta_0 = \int_0^T \sigma_s ds$*, and* $h(\Theta \mid \sigma_0)$ *is the risk neutral probability of* Θ_0 *which is conditional on* σ_0*.*

We see that the call option's true price is a weighted average of Black Scholes values. The correct hedge ratio is therefore

$$\frac{\partial C}{\partial \sigma_0} = \int_0^\infty \left[\frac{\partial f(\Theta_0)}{\partial \Theta_0} \frac{\partial \Theta_0}{\partial \sigma_0} h(\Theta_0 \mid \sigma_0) + f(\Theta_0) \frac{\partial h(\Theta_0 \mid \sigma_0)}{\partial \sigma_0} \right] d\Theta_0.$$

It is clear that this is not equal to the Black-Scholes vega, i.e. $\frac{\partial f(\sigma_0)}{\partial \sigma_0}$*.*

A simple example illustrates this absurdity even more clearly than the above use of vega hedging.

Example 9 ***Delta Hedging.***

This example shows how hedging a change in an option pricing model's constant parameter leads to nonsensical results.

Let us use a model where we assume that the stock price S *is non-random and (because it is riskless) that it grows at the risk free rate* r*. Then, the call option's value is*

$$\begin{aligned} c &= \widetilde{f}(S_t) \\ &= \max\{S_T - K, 0\} e^{-r(T-t)} \\ &= \max\{S_t - K e^{-r(T-t)}, 0\} \end{aligned}$$

where $\widetilde{f}(\cdot)$ *is the call option's value with strike price* K *and maturity* T*. We note that there are no risks in this call option model.*

Of course, we observe that the stock price S *is random, and that the model's price does not match market prices. Formally, the model is rejected.*

Nonetheless, we want to still use the model. Hence, we calibrate the model to the market price by selecting that stock price which equates the model's price to the market price. This yields an "implied stock price" (S_{imp}*). Of course,* S_{imp} *will not equal the market price (analogous to the implied volatility not equally the historical volatility in the Black-Scholes model). The calibrated option model now is transformed to a statistical model, only useful for pricing call options.*

But, let's still use this calibrated option model to hedge the stock price movements by "hedging the greek delta." We do this in the standard way by using a Taylor series expansion on the call's model price in ΔS *times the delta* $\frac{\partial \widetilde{f}}{\partial S}$*.*

It is easy to see that the delta in this Taylor series expansion is either 1 *or* 0 *depending on whether the stock price is in- or out-of-the-money, respectively. In words, one needs to either sell the entire stock or don't hedge. It is self-evident that this hedge will not work. This is true despite the fact that the calibrated option model is purposely constructed to provide reasonable option prices. Hedging the"greek" does not work here precisely because the original theoretical option model had no stock price risk embedded within it.*

4 How to Properly Use a Model

This section collects and summarizes the previous insights regarding how to properly use a risk management model. If followed, these simple rules will enable the model's user to make better decisions with the model than without.

Model Construction There are "good" and "bad" theoretical models. A good model's critical assumptions can be tested using observable market data, although its robust assumptions need not be. If the robust assumptions are not testable using observable market data, they should be intuitively valid. A bad model is one that does not satisfy these conditions. One should only use a good model.

Testing Models To test a theoretical model, one should test as many of the implications as possible, all of the critical assumptions, and be convinced that the robust assumptions are approximately true. All of these tests need to confirm the validity of the model. If any of these tests are rejected, the model is false and should not be used.

In addition, as time passes and new data is collected, the model should be continually retested for its validity. This is especially true of the critical assumptions, which if invalidated by an unexpected shift in the structure of the economy, cause the model's implications to no longer apply.

Calibration Calibration is useful if the theoretical model, based on parameters estimated with historical data, is rejected.[10] In this case, the calibrated model becomes a statistical model, used to estimate some quantity of interest. And, it should only be used for estimating that quantity. For example, if we are trying to determine the fair price of a derivative and calibration is used, then the calibrated model should only be used for estimating the derivative's price. Since the theoretical model is rejected, its other implications, including determining hedge ratios, are invalid.

This does not imply, however, that one cannot use a statistical model to determine hedge ratios. For example, one can run a simple regression

$$\Delta p_t = \alpha + \beta \Delta S_t + \varepsilon_t$$

where α, β are constants, p is the derivative's market price, S is the stock's price, and ε is the estimation error. The regression coefficient β is the estimated hedge ratio, suitable for use based on this statistical model. As estimated herein, the hedge ratio is based on patterns in the data and not a theoretical model. If the patterns change, the statistical hedging model will not longer work. This, of course, is the problem with statistical models.

[10] Of course, if the calibrated parameters equal the historical parameters, then the statement is factually correct, since the calibrated model will be identical to the theory model with historically estimated parameters. In this case, using the calibrated model as a theory model is also correct.

Hedging the Greeks When using a theoretical model, only the risks originally embedded within the model's construction can be hedged using the greeks - delta and gamma. The deterministic parameters of the model are not priced within the model, and using a Taylor series expansion to hedge their randomness is invalid. When using the Black-Scholes model, this applies to the underlying volatility - vega - and interest rates - rho. The need to hedge these risks invalidates the original theoretical model. A more general theoretical model need to be constructed, or a statistical hedging model needs to be employed.

5 Conclusion

The best way to understand models and their use is to consider an analogy. Models are analogous to medical prescription drugs. Prescription drugs have great medical benefits if used properly, with educated use. If used wrongly, however, prescription drugs can have negative consequences, even death. Because prescription drugs can cause death if used wrong, this does not mean that we should stop using them. It does mean, however that we need educated use. In fact, in reality, prescription drugs should probably be used more because they save and prolong lives. *The same is true of models.*

Financial markets have become too complex to navigate without risk management models. Determining a price - fair value - is not an issue because in many cases expert judgment can provide reasonable estimates. But,

- *there is no way to hedge a portfolio, i.e. determine hedge ratios, without a model,*
- *there is no way to determine the probability of a loss (risk measures) without a model, and*
- *there is no way to price a derivative in a illiquid market without a model.*

These issues are at the heart of risk management. Hence, financial risk management models are here to stay.

Models if used properly, help decision making. Models if used improperly, generate bad decisions and lead to losses. This doesn't mean that we should not use models. Quite the contrary. It only means that we need more educated use of models. Had models been properly used before the crisis, the 2007 credit crisis, perhaps the crisis would have not occurred.

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